Transpose of a Matrix

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$

then the transpose of A is denoted by A'. Which is matrix where the rows and columns have been interchanged.

that is;

$$A' = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \\ a_{14} & a_{24} & a_{34} \end{bmatrix}$$

This is sometimes also denoted by A^T.

Properties of Transpose of a Matrix

(i) Transpose of transpose of a matrix is the original matrix.

$$\boxed{(A')' = (A)}$$

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ then $A' = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}$ and

$$(A')' = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix}' = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix} = A$$

(ii) Transpose of the sum of matrices is the sum of the transposes of individual matrices.

Let $A = \begin{bmatrix} 2 & 4 \\ 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$

$$A+B = \begin{bmatrix} 3 & 4 \\ 6 & 4 \end{bmatrix} \text{ and } (A+B)' = \begin{bmatrix} 3 & 6 \\ 4 & 4 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 3 \\ 4 & 2 \end{bmatrix} \text{ and } B' = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

$$A' + B' = \begin{bmatrix} 3 & 6 \\ 4 & 4 \end{bmatrix} = (A+B)'$$

Hence in this case we verify that

$$\boxed{(A+B)' = A' + B'}$$

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(3) Transpose of a product of matrices is the product of the transposes of the matrices, taken in the reverse order.

$$\boxed{(AB)' = B'A'}$$

If $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$ then

$$AB = \begin{bmatrix} 2\times1 + 1\times2 & 2\times3+1\times2 \\ 4\times1+3\times2 & 4\times3+3\times2 \\ 1\times1+0\times2 & 1\times3+0\times2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 10 & 18 \\ 1 & 3 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 4 & 10 & 1 \\ 8 & 18 & 3 \end{bmatrix}$$

Again $B' = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ and $A' = \begin{bmatrix} 2 & 4 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

$$B'A' = \begin{bmatrix} 1\times2+2\times1 & 1\times4+2\times3 & 1\times1+2\times0 \\ 3\times2+2\times1 & 3\times4+2\times3 & 3\times1+2\times0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 10 & 1 \\ 8 & 18 & 3 \end{bmatrix}$$

Thus $(AB)' = (B'A')$

Trace of a matrix

The trace of a square matrix is the sum of its diagonal elements.

Thus in the matrix $\begin{bmatrix} 1 & 1 & 2 \\ 2 & 0 & 1 \\ 3 & 3 & -1 \end{bmatrix}$

The trace (tr) is $= 1+0+(-1)$

$= 0$

In general, $tr(A) = a_{11}+a_{22}+a_{33}+\dots+a_{nn}$

$$= \sum_{i=1}^{n} a_{ii}$$

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Idempotent matrix

A symmetric matrix that reproduces itself when multiplied by itself is termed as an idempotent matrix.

That is, A will be termed as idempotent if $AA = A$

* Show that the identity matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is idempotent matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since A is symmetric ; $A' = A$

* Show that the matrix

$$A = \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$ is an idempotent matrix

Here

$$\frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} \cdot \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 1+4+1 & -2-8-2 & 1+4+1 \\ -2-8-2 & 4+16+4 & -2-8-2 \\ 1+4+1 & -2-8-2 & 1+4+1 \end{bmatrix}$$

$$= \frac{1}{36}\begin{bmatrix} 6 & -12 & 6 \\ -12 & 24 & -12 \\ 6 & -12 & 6 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix} = A$$

Determinant of a square matrix

To each square matrix there corresponds a determinant. This determinant is written by enclosing the elements of the matrix by vertical bars.

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ is } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

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Value of a determinant

The value of a determinant can be expressed as a single number.

The second order determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

$\boxed{\text{Minors and Co-factors}}$

It is to be notified that a_1 is multiplied by a lower order determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ obtained by deleting the column and row containing a_1. Similarly, for the other terms, $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ is termed as minor of a_1 in the original matrix. There will be as many minors as the elements of the matrix.

Thus, the minor of b_2 will be obtained by deleting the 2nd row and 2nd column.

In $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, minor of $b_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

The Co-factor of an element is the minor of that element multiplied by $(-1)^{i+j}$ when ith row and jth column have been deleted since they contain the element.

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Now a_1 lies in the 1st row and 1st column, hence its cofactor is $(-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$. Similarly b_3 lies in 3rd row and 2nd column, hence its cofactor is $(-1)^{3+2}\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Thus $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 A_{11} + b_1 A_{12} + c_1 A_{13}$

where $A_{11} = \text{cofactor of } a_1 = (-1)^{1+1}\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

$A_{12} = \text{cofactor of } b_1 = (-1)^{1+2}\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

$A_{13} = \text{cofactor of } c_1 = (-1)^{1+3}\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

* Find the value of the determinant

$$\begin{vmatrix} 3 & 4 \\ 10 & -2 \end{vmatrix}$$

The value of determinant $(3)\times(-2) - (4)\times(10)$
$= -46$

Adjoint of a matrix

Let $A = (a_{ij})$ be any n square matrix, the transpose of the matrix (A_{ij}), where A_{ij} is the cofactor of a_{ij} in determinant A is called the Adjoint of A is written as Adj A.

Adjoint A = Adj $A = (A_{ij})'$ where A_{ij} is the cofactor of a_{ij} in $|A|$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \text{Adj } A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

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find the Adjoint of a Matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$a_{11} = 1$ and its cofactor $A_{11} = (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7$

$= (3\times2 - (-1)\times1) = 7$

$a_{12} = -2$ and its co-factor $A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -1 \\ -3 & 2 \end{vmatrix} = -1$

$a_{13} = 3$ and its cofactor $A_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 3 \\ -3 & 1 \end{vmatrix} = 11$

Similarly $A_{21} = 7$, $A_{22} = 11$, $A_{23} = 5$

$A_{31} = -7$, $A_{32} = 7$, $A_{33} = 7$

Adj. $A =$ Transpose of $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$

$=$ Transpose of $\begin{bmatrix} 7 & -1 & 11 \\ 7 & 11 & 5 \\ -7 & 7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 7 & 7 & -7 \\ -1 & 11 & 7 \\ 11 & 5 & 7 \end{bmatrix}$$

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The Inverse of a Matrix

Let A be any n square matrix. The n square matrix Matrix B is called inverse of A if $AB = BA = I_n$

The inverse of A is denoted by A^{-1} so $\boxed{B = A^{-1}}$

If B is the inverse of A, Then A is the inverse of B

$$A^{-1} \times A = AA^{-1} = I_n$$

∴ The necessary and sufficient condition for a square matrix to possess its inverse is that $|A| \neq 0$. If A and B are non-Singular square matrices of the same order then $(AB)^{-1} = B^{-1}A^{-1}$

The inverse of A is given by $A^{-1} = \frac{1}{|A|}(\text{Adj } A)$

Find A^{-1} if $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

$A_{11} = 2 \quad A_{12} = -1$

$A_{21} = -4 \quad A_{22} = 3$

Hence matrix of Co-factors is

$$\begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix}$$

Therefore $\text{Adj} \cdot A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$A^{-1} = \frac{\text{Adj} \cdot A}{|A|}$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1/2 & 3/2 \end{bmatrix}$$

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* Obtain the inverse of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{vmatrix} = 1(3\times12-5\times5) - 2(1\times12-1\times5) + 3(1\times5-1\times3)$$

$$= 1(36-25) - 2(12-5) + 3(5-3)$$

$$= 11 - 2(7) + 3(2) = 3$$

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 5 & 12 \end{vmatrix} = 11, \quad A_{12} = (-)\begin{vmatrix} 1 & 5 \\ 1 & 12 \end{vmatrix} = -7, \quad A_{13} = \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2$$

$$A_{21} = (-)\begin{vmatrix} 2 & 3 \\ 5 & 12 \end{vmatrix} = -9 \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 12 \end{vmatrix} = 9, \quad A_{23} = \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = 3$$

$$A_{31} = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 1, \quad A_{32} = (-)\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -2, \quad A_{33} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1$$

<u>matrix of co-factors</u>

$$= \begin{bmatrix} 11 & -7 & 2 \\ -9 & 9 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\text{Adj}.A = \begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{Adj}.A}{|A|}$$

$$= \frac{1}{3}\begin{bmatrix} 11 & -9 & 1 \\ -7 & 9 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{11}{3} & -3 & \frac{1}{3} \\ -7/3 & 3 & -2/3 \\ 2/3 & -1 & 1/3 \end{bmatrix}$$

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