Problems based on Determinants

Ex (1) Prove without expanding

$$\begin{vmatrix} 19 & 17 & 15 \\ 9 & 8 & 7 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

$$\Rightarrow \begin{vmatrix} 19 & 17 & 15 \\ 9 & 8 & 7 \\ 1 & 1 & 1 \end{vmatrix} \frac{\text{col}(1) - \text{Col.}(3)}{\text{Col.}(2) - \text{Col.}(3)} \begin{vmatrix} 4 & 2 & 15 \\ 2 & 1 & 7 \\ 0 & 0 & 1 \end{vmatrix}$$

Column (3) is kept as it is

Taking (2) common (3) is kept as it is form column(1)

$$\Rightarrow 2 \begin{vmatrix} 2 & 2 & 15 \\ 1 & 1 & 7 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

Ex-(2) Evaluate: $\begin{vmatrix} 2 & 45 & 55 \\ 1 & 29 & 32 \\ 3 & 68 & 87 \end{vmatrix}$

Expanding the given determinant by first column, we get

$$\Rightarrow 2 \begin{vmatrix} 29 & 32 \\ 68 & 87 \end{vmatrix} - 1 \begin{vmatrix} 45 & 55 \\ 68 & 87 \end{vmatrix} + 3 \begin{vmatrix} 45 & 55 \\ 29 & 32 \end{vmatrix}$$

$$\Rightarrow 2(347) - 1(175) + 3(-155)$$

$$= 694 - 175 - 465 = 54$$

Ex (3) Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \begin{matrix} Col.(3) - Col.(1) \\ Col.(2) - Col.(1) \end{matrix} \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix}$$

Expanding by row(1), we have

$$(1) \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-a)(c-a)(c-b)$$

$$= (a-b)(b-c)(c-a)$$

Ex-4 Find the value of

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Here Add row (2) and row (3) and subtract from row (1)

$$\begin{vmatrix} (b+c)-(b+c) & a-(2c+a) & a-(2b+a) \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & c & b \\ b & c+a & b \\ c & c & a+b \end{vmatrix} \frac{r(2) - r(1)}{r(3) - r(1)} = -2 \begin{vmatrix} 0 & c & b \\ b & a & 0 \\ c & 0 & a \end{vmatrix}$$

Interchange col(2) & col.(3) $2 \begin{vmatrix} 0 & b & c \\ b & 0 & a \\ c & a & 0 \end{vmatrix}$

$$= 2 \left[0 - b \begin{vmatrix} b & a \\ c & 0 \end{vmatrix} - c \begin{vmatrix} b & 0 \\ c & a \end{vmatrix} \right]$$

$$= 2 \left[-b \times -c - a + ba \right] = 4abC$$

Ex-(5) Evaluate $\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & b^2c & 0 \end{vmatrix}$

$$\Rightarrow \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & ab^2 & 0 \end{vmatrix} = abc \begin{vmatrix} 0 & b^2 & c^2 \\ a^2 & 0 & c^2 \\ a^2 & b^2 & 0 \end{vmatrix}$$

$$= abc\,(a^2b^2c^2) \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= a^3b^3c^3 \left[-1(0-1)+1(1-0)\right] = 2a^3b^3c^3$$

Ex-(6) Prove that $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

$$= 2(a+b+c) \left[(b+c+a)^2\right] = 2(a+b+c)^3$$

Rank of Matrix

Non - Singular and Singular Matrices

The square matrix A is called a Singular matrix if the Corresponding determinant $|A| = 0$

The square matrix $|A|$ is called a non Singular matrix if corresponding determinant $|A| \neq 0$

Rank of a matrix

The rank of matrix A denoted by $\rho(A)$ is equal to the order of the highest order non singular square matrix contained in A.

e.g:- The rank of matrix $\begin{pmatrix} 5 & 2 \\ -2 & -3 \end{pmatrix}$ is 2; because this second order matrix corresponds to the determinant $\begin{vmatrix} 5 & 2 \\ -2 & -3 \end{vmatrix}$ which is not zero, hence this given matrix Contains a non singular matrix of order 2.

e.g:- The rank of matrix $\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ is 1; the highest order determinant $\begin{vmatrix} 3 & 2 \\ 6 & 4 \end{vmatrix}$ is zero, next to this would be of the 1st order which is non-zero.

1. The rank of matrix is not related in any way to the number of zero elements in it.

2. The rank of matrix cannot exceed the number of its rows or number of its column - whichever is less.

3. The rank of a column matrix of any number of rows (matrix 3×1) is at the most 1; while rank of a matrix of 3×50 is at the most 3.

4. The rank of a matrix is at least one, unless the matrix has all zero elements i.e, when it is a null matrix. The rank of a matrix is zero.

5. Rank of transpose of a matrix A is the same as the rank of A.

Application of Matrices to the Solution of linear equations

Suppose we are required to solve the following set of Simultaneous equations for x_1, x_2 and x_3.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = h_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = h_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = h_3$$

Let us write the given equations in matrix form

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}$$

But since $\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Therefore, the given set of equations can be put as

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}}_{H}$$

Where A is 3×3 matrix

x is 3×1 matrix

and H is a 3×1 matrix.

This way, the three simultaneous given equations are written in the form of a single matrix equation as $Ax = H$.

We can find x_1, x_2 and x_3 if we write this single matrix equation in the form

$$\boxed{x = A^{-1}H}$$

(e.g) Solve the following equation by applying inverse method

$$2x_1 + 8x_2 = 34$$
$$4x_1 + 12x_2 = 56$$

Inverse method

$$\begin{bmatrix} 2 & 8 \\ 4 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 34 \\ 56 \end{bmatrix}$$

$$Ax = B$$
$$\boxed{x = A^{-1}B}$$

For inverse method we calculate co-factors $= [C_{ij}]$

Co-factors $C_{11} = 12$, $C_{12} = -4$, $C_{21} = -8$, $C_{22} = 2$

$$= \begin{bmatrix} 12 & -4 \\ -8 & 2 \end{bmatrix}$$

Value of determinant $= 2 \times 12 - 8 \times 4 = -8$

$$A^{-1} = \frac{\text{Adj matrix}}{|A|}$$

$$= \frac{\begin{bmatrix} 12 & -8 \\ -4 & 2 \end{bmatrix}}{-8}$$

$$= \begin{bmatrix} \frac{12}{-8} & \frac{-8}{-8} \\ \frac{-4}{-8} & \frac{2}{-8} \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{-3}{2} & 1 \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A^{-1} \cdot B$$

$$= \begin{bmatrix} -\frac{3}{2} & 1 \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix}_{2\times 2} \begin{bmatrix} 34 \\ 56 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{2} \times 34 + 1 \times 56 \\ \frac{1}{2} \times 34 - \frac{1}{4} \times 56 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} -51+56 \\ 17-14 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

$$\boxed{x_1 = 5,\ x_2 = 3}$$

The equilibrium condition for two substitute goods is given by

$$5P_1 - 2P_2 = 15$$
$$-P_1 - 8P_2 = 16$$

find the equilibrium prices.

$$\begin{bmatrix} 5 & -2 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 16 \end{bmatrix}$$

$$|A| = 5(8) - (-1)(-2) = 38$$

$$\text{Adj } A = \begin{bmatrix} 8 & 2 \\ 1 & 5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{38} \begin{bmatrix} 8 & 2 \\ 1 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{8}{38} & \frac{2}{38} \\ \frac{1}{38} & \frac{5}{38} \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{8}{38} & \frac{2}{38} \\ \frac{1}{38} & \frac{5}{38} \end{bmatrix} \begin{bmatrix} 15 \\ 16 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{120}{38} + \frac{32}{38} \\ \frac{15}{38} + \frac{80}{38} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{152}{38} \\ \frac{95}{38} \end{bmatrix}$$

$$\boxed{P_1 = 4,\ P_2 = 2.5}$$