Determinants

Determinant of a square matrix

To each square matrix there corresponds a determinant. This determinant is written by enclosing the elements of the matrix by vertical bars.

Thus the third order determinant of the matrix

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \text{ is } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Value of a determinant

The value of a determinant can be expressed as a single number.

The second order determinant

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

The third order determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$= a_1(b_2 c_3 - b_3 c_2) - b_1(a_2 c_3 - a_3 c_2) + c_1(a_2 b_3 - a_3 b_2)$$

Find the value of the determinant

$$\begin{bmatrix} 3 & 4 & 7 \\ 2 & 1 & 3 \\ 7 & 2 & 1 \end{bmatrix}$$

$$D = 3\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 2 & 1 \end{vmatrix} + 7\begin{vmatrix} 4 & 7 \\ 1 & 3 \end{vmatrix}$$

$$= 3(1-6) - 2(4-14) + 7(12-7) = -15 + 20 + 35 = 40$$

Properties of Determinants

Property - I

If all the rows of a determinant are changed into columns or vice versa, value of determinant remains unchanged.

i.e

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

If the matrix of original determinant is A then that of the changed one is transpose of A, i.e, A', hence $|A| = |A'|$

For e.g

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

Property - II

If any two adjacent rows (or columns) are interchanged, the sign of determinant changes.

For e.g

$$\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 0 \\ 1 & 0 & 3 \end{vmatrix} = (-)\begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$ → By interchanging Columns (1) and (2)

$$(-)\begin{vmatrix} 3 & 1 & 0 \\ 4 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = +\begin{vmatrix} 4 & 2 & 0 \\ 3 & 1 & 0 \\ 0 & 1 & 3 \end{vmatrix}$$ → By interchaning rows (1) and (2)

Property 3

If any column (or row) is shifted over any adjacent columns (or row) then the absolute value of the determinant does not change but its sign changes if the jumped columns (or rows) are odd in number.

$$\begin{vmatrix} 2 & 1 & -2 & 2 \\ 4 & 3 & 1 & 1 \\ 0 & 2 & 0 & 1 \\ 3 & 4 & 3 & 1 \end{vmatrix} = (-)^2 \begin{vmatrix} 1 & -2 & 2 & 2 \\ 3 & 1 & 4 & 1 \\ 2 & 0 & 0 & 1 \\ 4 & 3 & 3 & 1 \end{vmatrix}$$

The sign does not change because jumped columns are 2 in number.

$$= (-)^3 \begin{vmatrix} -2 & 2 & 2 & 1 \\ 1 & 4 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 3 & 3 & 1 & 4 \end{vmatrix}$$

→ Sign changes since jumped columns are 3 (odd)

Property 4

If there is any common factor in the elements of any one column or row, it can be taken out.

$$\begin{vmatrix} 3 & 1 & 2 \\ 6 & 2 & 2 \\ 12 & 3 & 4 \end{vmatrix} = 3 \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 2 \\ 4 & 3 & 4 \end{vmatrix}$$

∴ 3 is common factor in the first column

$$= 2 \times 3 \begin{vmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 4 & 3 & 4 \end{vmatrix}$$

∴ 2 is common factor in the 2nd row.

Property 5

If all the elements of any one row (or column) are multiplied (or divided) by K, then value of the determinant is also multiplied (or divided) by K. This property follows from property 4.

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$ Multiply first Column by 2, we have

$$\begin{vmatrix} 2 & 2 & 3 \\ 4 & 0 & 1 \\ 6 & 4 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 4 & 2 \end{vmatrix}$$

Property-6

If any two columns (or any two rows) of a determinant are exactly identical, the value of the determinant is zero.

For e.g $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 0$$ [Since row 1 and 3 are identical]

and $$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 1 & 2 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 1 & 4 \end{vmatrix} = 0$$

[Columns 1 and 2 are identical]

Property-7

If any Column (or row) or a multiple of any column (or row) is added or subtracted from any other column (or row) the value of the determinant remains unchanged.

Let

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 6 & 2 & 12 \end{vmatrix}$$ multiply Column (1) by 2 and subtract from Column (2)

$$\begin{vmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & -4 & 0 \\ 6 & 2 & -12 & 12 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 0 & 0 \\ 6 & -10 & 12 \end{vmatrix}$$ (Multiply Col. 1 by 3 and Subtract from Col. 3)

Now expanding by first row, we have

$$|A| = 1 \begin{vmatrix} 0 & -6 \\ -10 & -6 \end{vmatrix} + 0 + 0$$

$$= 1(0-60) = -60$$

Also directly expanding by the first row

$$|A| = 1 \begin{vmatrix} 4 & 0 \\ 2 & 12 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 6 & 12 \end{vmatrix} + 3 \begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix}$$

$$= 1(48-0) - 2(24-0) + 3(4-24)$$

$$= 48 - 48 - 60 = -60$$

Property-8

If a determinant is of the form

$$\begin{vmatrix} a_1+b_1 & c_1 & d_1 \\ a_2+b_2 & c_2 & d_2 \\ a_3+b_3 & c_3 & d_3 \end{vmatrix}$$ it can be written as the sum of two determinants.

$$\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

For. eg

$$\begin{vmatrix} 4 & 2 & 1 \\ 7 & 3 & 2 \\ 10 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 3+1 & 2 & 1 \\ 5+2 & 3 & 2 \\ 7+3 & 4 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 2 \\ 7 & 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 2+1 & 2 & 1 \\ 3+2 & 3 & 2 \\ 4+3 & 4 & 3 \end{vmatrix} + \text{zero}$$

$$= \begin{vmatrix} 2 & 2 & 1 \\ 3 & 3 & 2 \\ 4 & 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 3 & 4 & 3 \end{vmatrix}$$

$$= 0 + 0$$

$$= 0$$ [The columns are identical)