Notes of Maths Sem-II (Core-04)

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Exact Differential Equation

Definition: An Equation $Mdx+Ndy=0$ is said to be 'exact' when M and N are of the forms $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ respectively.

Then form $\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy=0$ is equivalent to $du=0$

on integration, it gives $u=C$

where C is one arbitrary Constant. Every equation of this type

(1) is either exact or Can be made exact by a suitable Integrating factor the number of such factor's is unlimited.

— x —

Q.1 Solve $ydx-xdy=xydx$

Sol:. Dividing by xy, $\frac{dx}{x}-\frac{dy}{y}=dx$

Integrating

$\log x-\log y=x+C$ C being the Constant of Integration

or $\log \frac{x}{y}=x+c$

$\therefore \frac{x}{y}=e^{x+c}$

$\therefore x=ye^{x+c}=Aye^{x}$ where $e^{C}=A$

— x —

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Q. Solve $xy\frac{dy}{dx} = \frac{1+y^2}{1+x^2}(1+x+x^2)$

We have $\frac{y\,dy}{1+y^2} = \frac{1+x+x^2}{(1+x^2)x}\,dx$

or $\frac{1}{2}\times\frac{2y}{1+y^2} = \left(\frac{1}{x} + \frac{1}{1+x^2}\right)dx$

Integrating

$\therefore \frac{1}{2}\log|1+y^2| = \log x + \tan^{-1}x + c$

$\therefore \log\sqrt{(1+y^2)} = \log x + \tan^{-1}x + c$

$\therefore \log\frac{\sqrt{1+y^2}}{x} = \tan^{-1}x + c$

$\therefore \sqrt{1+y^2} = xe^{\tan^{-1}x + c}$

$\therefore 1+y^2 = x^2 e^{2\tan^{-1}x} e^{2c}$

$= Ax^2 e^{2\tan^{-1}x}$ $(e^{2c} = A)$

Q. Solve $\frac{dy}{dx} + \frac{y}{x} = x^2$ when $y=1$, $x=1$

Sol:- We have $x\,dy + y\,dx = x^3\,dx$

Integrating, we get →

$xy = \frac{x^4}{4} + K$

When $x=1$, $y=1$

$\therefore 1 = \frac{1}{4} + K$ or $K = \frac{3}{4}$

Hence $xy = \frac{x^4}{4} + \frac{3}{4}$

Q. Solve $(1+xy)y\,dx + (1-xy)x\,dy = 0$

Sol:- We have $y\,dx + x\,dy + xy^2\,dx - x^2y\,dx = 0$

Dividing by x^2y^2

$\frac{xy\,dy + y\,dx}{x^2y^2} + \frac{dx}{x} - \frac{dy}{y} = 0$

or $\frac{d(xy)}{(xy)^2} + \frac{dx}{x} - \frac{dy}{y} = 0$

Integrating $-\frac{1}{xy} + \log x - \log y = K$

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CLAIRAUT'S Equation and Singular Solution.

The Equation $y = px + f(p)$ ---- (1) where $p = \frac{dy}{dx}$ is known as Clairaut's Equation.

Q:- Solve $y = px + p^2$

Sol. we have $y = px + p^2$ ---- (1)

It is of form $y = px + f(p)$. So diff- w.r.to x

$$\frac{dy}{dx} = \text{i.e } p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}$$

$$\text{i.e } \frac{dp}{dx}(x + 2p) = 0$$

$\therefore$ either $\frac{dp}{dx} = 0$ or $x + 2p = 0$

when $\frac{dp}{dx} = 0$

i.e. $dp = 0$

Then Integrating

$$p = c$$

Putting the value of p in (1) $y = cx + c^2$ ---

this is the general Solution.

Again when $x + 2p = 0$

$$\text{i.e } p = -\tfrac{1}{2}x$$

Then Putting the value of p in (1)

$$y = -\tfrac{1}{2}x^2 + \tfrac{1}{4}x^2 \text{ ---}$$

i.e $4y + x^2 = 0$ this is the Singular Solution.

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Q. Solve $p + \cos px \sin y = \sin px \cdot \cos y$

Sol: We have $p = \sin px \cdot \cos y - \cos px \cdot \sin y$

$\Rightarrow p = \sin(px - y)$

$\Rightarrow \sin^{-1} p = px - y$

$\Rightarrow y = px - \sin^{-1} p$ (This is in the form of Clairaut) — (1)

Diff —

$\frac{dy}{dx} = p + x\frac{dp}{dx} - \frac{1}{\sqrt{1-p^2}}\frac{dp}{dx}$

$\Rightarrow p = p + \frac{dp}{dx}\left(x - \frac{1}{\sqrt{1-p^2}}\right)$

$\Rightarrow \frac{dp}{dx}\left(x - \frac{1}{\sqrt{1-p^2}}\right) = 0$

$\therefore$ either $\frac{dp}{dx} = 0$ or $x - \frac{1}{\sqrt{1-p^2}} = 0$

When $\frac{dp}{dx} = 0$ i.e. $dp = 0$.

Integrating

$p = c$

Putting the value of p in (1)

$y = cx - \sin^{-1} c$

Again when $x = \frac{1}{\sqrt{1-p^2}}$ i.e. $1 - p^2 = \frac{1}{x^2}$

i.e. $p^2 = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2}$

i.e. $p = \frac{\sqrt{x^2-1}}{x}$

Putting the value of p in (1)

$y = \sqrt{x^2-1} - \sin^{-1}\left(\frac{\sqrt{x^2-1}}{x}\right)$

This is Singular Solution

— x —

Q. Solve $(y-1)p - xp^2 + 2 = 0$

We have $yp = p - 2 + p^2x = p^2x + (p-2)$

or $y = px + (1 - 2/p)$ — ①

which is in Clairaut's form.

Diff — ① w.r.to x

$$\frac{dy}{dx} = p = p + x\frac{dp}{dx} + \frac{2}{p^2}\frac{dp}{dx}$$

or $\frac{dp}{dx}\left(x + \frac{2}{p^2}\right) = 0$

∴ either $\frac{dp}{dx} = 0$ or $x + \frac{2}{p^2} = 0$

when $\frac{dp}{dx} = 0$ i.e $dp = 0$

Integrating

$p = c$

∴ the general solution is $y = cx + 1 - 2/c$

Also when $x + 2/p^2 = 0$, $xp^2 + 2 = 0$ — ②

The given Eq becomes

$(y-1)p + 4 = 0$

or $p = \frac{4}{1-y}$

Putting the value of p in ② we get the Singular solution,

$x \cdot \frac{16}{(1-y)^2} + 2 = 0$

or $8x + (1-y^2) = 0$

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Equation Solvable for y:

These Equation are of the form $y = f(x,p)$
Differentiating w.r.to x, $p = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial p}\cdot\frac{dp}{dx}$ which
Can be solved in the form $\phi(x,p,c) = 0$
Eliminate p between $y = f(x,p)$ and $\phi(x,p,c) = 0$
then the resulting eliminant is the general
Solution of the equation.

Q. Solve $y = x\left[\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3\right]$

Solution:- We have $y = x(p+p^3)$ ← (1)

Diff (1) w.r.to x

$$\frac{dy}{dx} = p = p + p^3 + x(1+3p^2)\frac{dp}{dx}$$

$$\text{or } -p^3 = x(1+3p^2)\frac{dp}{dx}$$

$$\text{or } \frac{dx}{x} = \frac{1+3p^2}{-p^3}dp$$

$$= -\frac{1}{p^3}dp - \frac{3}{p}dp$$

Integrating, we get—

$$\log x = \frac{1}{2p^2} - 3\log p - \log k$$

$$\text{or } \log(kxp^3) = \frac{1}{2p^2}$$

$$\text{or } kxp^3 = e^{\frac{1}{2p^2}} \quad \text{—— (2)}$$

The p-eliminant of (1) and (2) is
the required Solution.

———— ✗ ————

Q. Solve $y = 2px + 4xp^2$

Solution:- Differentiating the given equation with respect to x, we get —

$$\frac{dy}{dx} = p = 2p + 2x\frac{dp}{dx} + 4p^2 + 8xp\frac{dp}{dx}$$

$$\text{or } 2x\frac{dp}{dx}(1+4p) = -p - 4p^2 = -p(1+4p)$$

$$\text{or } (1+4p)\left[2x\frac{dp}{dx} + p\right] = 0$$

$$\therefore 1+4p = 0 \quad \text{or} \quad 2x\frac{dp}{dx} + p = 0$$

When $1+4p = 0$ or $1 + 4\frac{dy}{dx} = 0$

$$\text{or } dx + 4dy = 0$$

Integrating, we get —

$$x + 4y = k$$

and, when $2x\frac{dp}{dx} + p = 0$

$$\text{or } 2\frac{dp}{p} + \frac{dx}{x} = 0$$

Integrating we get —

$$2\log p + \log x = \log k.$$

$$\text{or } p^2 x = k \quad \text{or } p = \sqrt{k/x}$$

Putting the value of p in the given eq. we get — $y = 2x\sqrt{k/x} + 4x \times k/x$

$$\text{or } (y-4k)^2 = 4kx$$

Hence the Solution is

$$(y-4k)^2 = 4kx$$

Q. Solve $y = x(p^2 - 2p + 2)$

Sol: We have $y = x(p^2 - 2p + 2)$

It is of the form

$y = f(x, p)$ Diff. w.r.t. x

$$\frac{dy}{dx} = p^2 - 2p + 2 + x\left(2p\frac{dp}{dx} - 2\frac{dp}{dx}\right)$$

i.e. $-p^2 + 3p - 2 = 2x\frac{dp}{dx}(p-1)$

i.e. $-(p-1)(p-2) - 2x\frac{dp}{dx}(p-1) = 0$

i.e. $(p-1)\left[(p-2) + 2x\frac{dp}{dx}\right] = 0$

When $p - 1 = 0$, i.e. $p = 1$

Putting the value of p in (1) we get $y = x$

or, when $p - 2 + 2x\frac{dp}{dx} = 0$

i.e. $\frac{dx}{x} + 2\frac{dp}{p-2} = 0$

Integrating,

$$\log x + 2\log(p-2) = \log k^2$$

i.e. $\log[x(p-2)^2] = \log k^2$

i.e. $x(p-2)^2 = k^2$

i.e. $p - 2 = \frac{k}{\sqrt{x}}$

i.e. $p = \frac{k + 2\sqrt{x}}{\sqrt{x}}$

Putting the value of p in (1)

$$y = x \cdot \frac{k + 2\sqrt{x}}{\sqrt{x}} \cdot \frac{k}{\sqrt{x}} + 2$$

i.e. $y = k(k + 2\sqrt{x}) + 2x$

———— ✗ ————

Equation Solvable for x

The Equations are of the form $x = f(y,p)$

Diff. w.r.to y, $\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{dp}{dy}$ which

Can be solved in the form $\phi(y,p,c)=0$

Eliminate p between $x = f(y,p)$ and $\phi(y,p,c)=0$ and the resulting eliminant is the general Solution of the equation.

Q. Solve $p^2y + 2px = y$

We have $x = \frac{y}{2p} - \frac{yp}{2}$

Diff. w.r.to y, we get —

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\text{or, } \frac{1}{2p} + \frac{p}{2} = -\frac{y}{2}\frac{dp}{dy}\left(\frac{1}{p^2}+1\right)$$

$$\text{or } \frac{1+p^2}{2p} = -\frac{y}{2}\frac{dp}{dy}\frac{1+p^2}{p^2}$$

$$\text{or } 1 = -\frac{y}{p}\frac{dp}{dy} \text{ as } 1+p^2 \neq 0$$

$$\text{or } p\,dy + y\,dp = 0 \text{ or } d(py) = 0$$

Integrating, we get —

$$py = K, \text{ or } p = K/y$$

Putting the value of p in the given Equation. we get →

$$y\cdot\frac{K^2}{y^2} + 2x\frac{K}{y} = y$$

$$\text{or } K^2 + 2Kx = y^2$$

which is the required Solution.

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It is clairaut form $x = f(y,p)$

diff. w.r.to y

$$\frac{dx}{dy} = p + y\frac{dp}{dy} + 2ap\frac{dp}{dy}$$

$$= \frac{1}{p} - p = \frac{dp}{dy}(y + 2ap)$$

i.e. $$\frac{dy}{dp} = \frac{py}{1-p^2} + \frac{2ap^2}{1-p^2}$$

i.e. $$\frac{dy}{dp} - \frac{p}{1-p^2}\cdot y = \frac{2ap^2}{1-p^2} \quad \text{---(2)}$$

It is linear Equation

Its I.F $= e^{-\int \frac{p\,dp}{1-p^2}} = e^{\frac{1}{2}\log(1-p^2)}$

$= e^{\log(\sqrt{1-p^2})} = \sqrt{1-p^2}$

multiplying (2) by I.F on integrating

$$y\sqrt{1-p^2} = 2a\int \frac{p^2}{1-p^2} \times \sqrt{1-p^2}\,dp$$

$$= 2a\int \frac{p^2}{\sqrt{1-p^2}}dp = -2a\int \frac{(1-p^2)-1}{\sqrt{1-p^2}}dp$$

$$= -2a\int \sqrt{1-p^2}\,dp + 2a\int \frac{dp}{\sqrt{1-p^2}}$$

$$= -2a\left[\frac{p}{2}\sqrt{1-p^2} + \frac{1}{2}\sin^{-1}p\right] + 2a\sin^{-1}p + K$$

i.e. $$y\sqrt{1-p^2} = -ap\sqrt{1-p^2} + a\sin^{-1}p + K \quad \text{---(3)}$$

By eliminating p from (1) & (3) we get the required Solution

---x---

Q. Solve $y = 2px + y^2p^3$

Sol:- we have $2x = \frac{y}{p} - y^2p^2$

Diff- w.r.to y, we get-

$$2\frac{dx}{dy} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - 2py^2\frac{dp}{dy} - 2p^2y$$

or $\frac{2}{p} - \frac{1}{p} + 2p^2y + \frac{y}{p}\frac{dp}{dy}\left(\frac{1}{p} + 2p^2y\right) = 0$

or $\left(\frac{1}{p} + 2p^2y\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$

or $1 + \frac{y}{p}\frac{dp}{dy} = 0$, or $\frac{dp}{p} + \frac{dy}{y} = 0$ ...(1)

or $\log p + \log y = \log c$ or $p = c/y$ ...(2)

Put $p = c/y$ in to given equation, we get the solution $y^2 = 2cx + c^3$.

Q. Solve $x = y - p^2$ where $p = \frac{dy}{dx}$

Sol.- Diff- w.r.to y

$$\frac{1}{p} = 1 - 2p\frac{dx}{dp} \text{ or } 2p\frac{dx}{dp} = 1 - \frac{1}{p} = \frac{p-1}{p}$$

or $2dx = \frac{p-1}{p^2}dp$

$\therefore 2x = \int\left(\frac{1}{p} - \frac{1}{p^2}\right)dp$

$= \log p + \frac{1}{p} + c$

Also from the given eq. $p = \sqrt{y-x}$

$\therefore$ the p-eliminate is $2x = \log\sqrt{y-x} + \frac{1}{\sqrt{y-x}} + c$.

Q. Solve $p^3 - p(y+3) + x = 0$

Sol:- We have $x = p(y+3) - p^3$ — (1)

It is of the form $x = f(y, p)$

diff. w.r.to y

$\frac{dx}{dy}$ i.e $\frac{1}{p} = \frac{dp}{dy}(y+3) + p - 3p^2\frac{dp}{dy}$

i.e $\frac{1-p^2}{p} = \frac{dp}{dy}(y+3) - 3p^2\frac{dp}{dy}$

i.e $\frac{dy}{dp} = \frac{yp}{1-p^2} + 3(1-p^2)\cdot\frac{p}{1-p^2}$

i.e $\frac{dy}{dp} - \frac{p}{1-p^2}\cdot y = 3p$ — (2)

It is a linear Equation. IF $= e^{-\int \frac{p}{1-p^2} dp}$

$= e^{\frac{1}{2}\log(1-p^2)} = e^{\log\sqrt{1-p^2}} = \sqrt{1-p^2}$

multiplying (2) by I.F and Integrating, we get

$y\sqrt{1-p^2} = 3\int p\sqrt{1-p^2}\,dp$

$= -3\int t^2\,dt$ | Put $1-p^2 = t^2$

$= -t^3 + k$ | $-2p\,dp = 2t\,dt$

or $y\sqrt{1-p^2} = -(1-p^2)^{3/2} + k$ — (3)

Eliminating p from (1) and (3) we get the required Solution.

Q. Solve $x\frac{dy}{dx} = 1 + \left(\frac{dy}{dx}\right)^2$

Sol:- We have $xp = 1 + p^2$ or $x = \frac{1}{p} + p$

This is Solvable for x,

diff. w.r.to y, we get

$\frac{dx}{dy} = \frac{1}{p} = -\frac{1}{p^2}\frac{dp}{dy} + \frac{dp}{dy}$

or $\frac{dy}{dp}\times\frac{1}{p} = -\frac{1}{p^2} + 1$

or $\frac{dy}{dp} = p - \frac{1}{p}$

$dy = p\,dp - \frac{dp}{p}$

Integrating, we get

$y = \frac{p^2}{2} - \log p + k$.

The P eliminant between this Eqn and the given Equation is the required Solution.

Linear Differential Eq of second order with Constant Co-efficients.

Introduction: — Let us Consider a general linear Equation of the second order.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = V \quad \text{---} \quad (1)$$

where P and Q are Constant and V is a given function of x the Complete Solution of (1) may be written as $y = u + w$.

Q. Solve $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$

Sol:- The given equation is $(D^3 + 3D^2 + 2D)y = x^2$

For C.F $D^3 + 3D^2 + 2D = 0$

or $D(D^2 + 3D + 2) = 0$

or $D(D+1)(D+2) = 0$

$\therefore D = 0, -1, -2$

$\therefore$ C.F $= C_1 + C_2 e^{-x} + C_3 e^{-2x}$.

$$P.I = \frac{1}{D(D^2+3D+2)} x^2$$

$$= \frac{1}{2D}\left(1 + \frac{D^2+3D}{2}\right)^{-1} x^2$$

$$= \frac{1}{2D}\left[1 - \frac{D^2+3D}{2} + \frac{(D^2+3D)^2}{4} - \cdots\right] x^2$$

$$= \frac{1}{2D}\left(x^2 - \frac{2+6x}{2} + 9/2\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - x - \frac{3x^2}{2} + \frac{9}{2}x\right]$$

$$= \frac{1}{12}\left[2x^3 - 9x^2 + 21x\right]$$

$\therefore$ The Complete Solution = C.F + P.I

$$= C_1 + C_2 e^{-x} + C_3 e^{-2x} + \frac{1}{12}\left[2x^3 - 9x^2 + 21x\right]$$

Q. Solve $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x^2$

Sol:- we have $(D^2+D+1)y = x^2$

∴ the auxiliary equation is $m^2+m+1=0$

$$\therefore m = \frac{-1 \pm \sqrt{3}i}{2}$$

∴ C.F = Complementary function

$$= e^{-x/2}\left(A\cos\sqrt{3}/2\,x + B\sin\sqrt{3}/2\,x\right)$$

Now P.I $= \frac{1}{D^2+D+1}x^2$

$$= \frac{D-1}{D^3-1}x^2 = -(D-1)(1-D^3)^{-1}x^2$$

$$= -\left[(D-1)(1+D^3+D^6+D^9+\cdots)\right]x^2$$

$$= -\left[D-1+D^4-D^3+\cdots\right]x^2$$

$$= x^2 - 2x$$

∴ the Complete Solution is

$$y = e^{-\frac{x}{2}}\left(A\cos\sqrt{3}/2\,x + B\sin\sqrt{3}/2\,x\right) + x^2 - 2x.$$

Q: Integrate $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{2x}$

Sol:- the given equation is $D^2+D+1=0$

$$\text{or } D^2+D+1/4 = -1+\frac{1}{4}$$

$$\text{or } \left(D+\frac{1}{2}\right)^2 = \frac{3i^2}{4} \text{ or } D+1/2 = \pm\frac{\sqrt{3}i}{2}$$

$$\text{or } D = -1/2 \pm \frac{\sqrt{3}i}{2} = \frac{-1\pm\sqrt{3}i}{2}$$

$$\therefore C.F = C_1 e^{\frac{-1+\sqrt{3}i}{2}x} + C_2 e^{\frac{-1-\sqrt{3}i}{2}x}$$

$$= C_1 e^{-\frac{x}{2}} e^{\frac{\sqrt{3}i}{2}x} + C_2 e^{-x/2} e^{-\frac{\sqrt{3}i}{2}x}$$

$$= e^{-x/2}\left[C_1\left(C_3\sqrt{3}/2\,x + i\sin\sqrt{3}/2\,x\right) + C_2\left(C_3\sqrt{3}/2\,x - i\sin\sqrt{3}/2\,x\right)\right]$$

$$= e^{-x/2}\left[C_3\sqrt{3}/2\,x\,(C_1+C_2) + \sin\sqrt{3}/2\,x\,(C_1 i - C_2 i)\right]$$

Let $C_1+C_2 = C\sin\alpha$, $C_1 i - C_2 i = C\cos\alpha$

$$C.F = Ce^{-x/2}\sin(\alpha + \sqrt{3}/2\,x)$$

$$P.I = \frac{1}{D^2+D+1}e^{2x} = e^{2x}\cdot\frac{1}{4+2+1} = \frac{1}{7}e^{2x}$$

Hence Complete Sol:- $Ce^{-x/2}\sin(\alpha + \sqrt{3}/2\,x) + \frac{1}{7}e^{2x}$

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The auxilliary equation is $m^2-4m-5=0$

or $(m-5)(m+1)=0 \therefore m=5,-1$

∴ Complementary function $= C_1 e^{5x} + C_2 e^{-x}$

Now the P.I $= \frac{1}{D^2-4D-5} \sin x$

$= \frac{1}{-1-4D-5} \sin x = \frac{1}{-2(3+2D)} \sin x$

$= \frac{3-2D}{-2(9-4D^2)} \sin x$

$= \frac{3\sin x - 2\cos x}{-2(9+4)}$

∴ The Complete Solution is $y = C_1 e^{5x} + C_2 e^{-x} + \frac{3\sin x - 2\cos x}{-26}$

Solve $\frac{d^2y}{dx^2} - y = \cos 2x$

Sol:- We have $(D^2-1)y = \cos 2x$

∴ The auxilliary equation is $m^2-1=0$

$\therefore m = \pm 1$

∴ The C.F $= C_1 e^{x} + C_2 e^{-x}$

Now the P.I $= \frac{1}{D^2-1} \cos 2x = \frac{1}{-2^2-1} \cos 2x = -\frac{1}{5} \cos 2x$

∴ The Complete Solution is

$y = C_1 e^{x} + C_2 e^{-x} - \frac{1}{5} \cos 2x$

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Q. Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x + e^{mx}$

Sol: The given equation is $(D^2 - 5D + 6)y = x + e^{mx}$

For C.F $D^2 - 5D + 6 = 0$ or $(D-2)(D-3) = 0$

$\therefore D = 2, 3$

$\therefore$ C.F $= C_1 e^{2x} + C_2 e^{3x}$

P.I $= \frac{1}{D^2 - 5D + 6} x + \frac{1}{D^2 - 5D + 6} e^{mx}$

$= \frac{1}{6}\left(1 + \frac{D^2 - 5D}{6}\right)^{-1} x + \frac{e^{mx}}{m^2 - 5m + 6}$

$= \frac{1}{6}\left(1 - \frac{D^2 - 5D}{6} + \cdots\right) x + \frac{e^{mx}}{m^2 - 5m + 6}$

$= \frac{1}{6}\left(x + \frac{5}{6}\right) + \frac{e^{mx}}{m^2 - 5m + 6}$

$\therefore$ The Complete Solution = C.F + P.I

$= C_1 e^{2x} + C_2 e^{3x} + x/6 + \frac{5}{36} + \frac{e^{mx}}{m^2 - 5m + 6}$

— ✕ —

Q. Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + C_3 x$

Solution, The given equation is $(D^3 - 3D^2 + 4D - 2)y = e^x + C_3 x$

For C.F, $D^3 - 3D^2 + 4D - 2 = 0$

or $D^3 - D^2 - 2D^2 + 2D + 2D - 2 = 0$

$= D^2(D-1) - 2D(D-1) + 2(D-1) = 0$

or $(D-1)(D^2 - 2D + 2) = 0$

or $D = 1$, $D^2 - 2D + 1 = -1$

$\therefore (D-1)^2 = i^2$ or $D - 1 = \pm i$, or $D = 1 \pm i$

$\therefore$ C.F $= C_1 e^x + C_2 e^{(1+i)x} + C_3 e^{(1-i)x}$

$= C_1 e^x + C_2 e^x e^{ix} + C_3 e^x e^{-ix}$

$= C_1 e^x + e^x [C_2(\cos x + i \sin x) + C_3(\cos x - i \sin x)]$

$= C_1 e^x + e^x [(C_2 + C_3)\cos x + (C_2 i - C_3 i)\sin x]$

Let $C_2 + C_3 = C\sin\alpha$ or $C_2 i - C_3 i = C\cos\alpha$

C.F $= C_1 e^x + C e^x \sin(x + \alpha)$

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$\frac{1}{D}(1+D^2)^{-1} x^0 + \frac{-3\sin x - \cos x}{-9-1}$

$\frac{1}{D}(1-D^2+\cdots)x^0 + \frac{1}{10}(3\sin x + \cos x)$

$xe^x + \frac{1}{10}(3\sin x + \cos x)$

Complete Solution = C.F + P.F

$= C_1 e^x + C_2 e^x \sin(x+\alpha) + xe^x + \frac{1}{10}(3\sin x + \cos x)$

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Solve $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$

The given Eq is $(D^4 + 2D^2 + 1)y = x^2 \cos x$

C.F $D^4 + 2D^2 + 1 = 0$

or $(D^2+1)^2 = 0$

or $D^2 = -1, -1 = i^2, i^2$

$D = \pm i, \pm i$

$CF = (C_1 + C_2 x)e^{ix} + (C_3 + C_4 x)e^{-ix}$

$= (C_1 + C_2 x)(\cos x + i\sin x) + (C_3 + C_4 x)(\cos x - i\sin x)$

$= \cos x[(C_1 + C_3) + x(C_2 + C_4)] + \sin x[(C_1 i - C_3 i) + x(C_2 i - C_4 i)]$

$C_1 + C_3 = A, \quad C_2 + C_4 = B, \quad C_1 i - C_3 i = C$

$C_2 i - C_4 i = D$

$C.F = (A + Bx)\cos x + (C + Dx)\sin x$

$P.I = \frac{1}{(D^2+1)^2} x^2 \cos x$

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$= \text{Real part of } \frac{1}{(D^2+1)^2} x^2 e^{ix}$

$= \text{R.P. of } e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2$

$= \text{R.P. of } e^{ix} \frac{1}{(D^2+2Di)^2} x^2$

$= \text{R.P. of } e^{ix} \frac{1}{-4D^2}\left(1+\frac{D}{2i}\right)^{-2} x^2$

$= \text{R.P of } e^{ix} \frac{1}{-4D^2}\left(1-\frac{D}{i}+\frac{3}{4}\frac{D^2}{-1}+\dots\right)x^2$

$= \text{R.P of } e^{ix} \frac{1}{-4D^2}\left(x^2+2xi-\frac{3}{2}\right)$

$= -\frac{1}{4}\text{ R.P. of } (\cos x+i\sin x)\left[\frac{x^4}{12}+\frac{ix^2}{3}-\frac{3}{4}x^2\right)$

$= -\frac{1}{4}\left[\frac{1}{12}x^4\cos x-\frac{3}{4}x^2\cos x-\frac{1}{3}x^2\sin x\right)$

∴ The Complete Solution = C.F + P.I

$= (A+Bx)\cos x+(C+Dx)\sin x+\frac{1}{12}x^2\sin x+\frac{1}{48}(9x^2-x^4)\cos x$

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